THERMAL REACTION OF VISCOELASTIC BODIES TO THERMAL IMPACT ON THE BASIS OF A NEW EQUATION OF DYNAMIC THERMOVISCOELASTICITY

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A dynamic thermoviscoelasticity equation is proposed for viscoelastic bodies described by the Maxwell model. New dynamic thermoviscoelasticity problems are examined that generalize known solutions of thermomechanics within the framework of classical Fourier phenomenology of heat propagation in solids.

The urgency of problems of intensive thermal flux interaction with solid bodies has been elevated in recent decades in connection with the production of powerful emitters. The thermal action of a plasma flux, laser or electron beams is utilized in different processes of material treatment by concentrated energy fluxes. Conditions are produced for the jump-like change in the surface temperature of the solid body or the medium bounding it (the so-called thermal impact), that results in the appearance of powerful thermal stress waves in the bodies that is sufficient for crack formation. An urgent problem of estimating the role of temperature fields and thermoelastic waves in the mechanism of thermal fracture of solids occurs within the terminology of corresponding dynamic thermomechanics problems (classical [1-3] and generalized [4]).

Many papers have appeared on this subject. The physical regularities of the thermal stress state have been studied in elastic bodies within the framework of classical Fourier phenomenology on heat propagation in solids (see the survey [5] where systematic classification of publications from the first up to 1987 is given). Analogous studies have been performed in more complex cases of generalized thermomechanics [4, 6] with the A. V. Lykov hypothesis of the finite rate of heat propagation [7] taken into account in anisotropic and isotropic bodies, within the framework of the linearized theory of thermoelasticity with thermal memory take into account [8], within the terminology of generalized magnetothermoelasticity [9-11]. The thermelastic state of an isotropic body is investigated most completely. For this case we present the governing relationships and we obtain the fundamental equation of dynamic thermoelasticity in stresses in general form.

Let $\mathscr{D}$ be a finite or partially bounded domain of variation of the space variables (x, y, z), respectively, of the body geometry and dimension in which the thermoelasticity process is studied, let S be the boundary of the domain that can be considered a piecewise smooth surface, and T_0 the body temperature in the initial unstrained and unstressed state (there are no external forces). The domain $\mathscr{D}$ will be deformed because of the action of thermal sources and external heating (or cooling), and its temperature T(x, y, z, t) = T(M, t) will vary. Displacements $\bar{U}(M, t) = \{U_x(M, T), U_y(M, t), U_z(M, t)\}$, strains $\varepsilon_{ik}(M, t)$, and stresses $\sigma_{ik}(M, t)$ (i, k = x, y, z) will occur in the body.

The thermoelastic state of the elastic body is described by a system of equations (in the subscript notation) into which there will enter [1, 2]

the equilibrium equations

$$\sigma_{ik,k}(M, t) = \rho \ddot{U}_i(M, t), \ M \in \mathscr{D}, \ t > 0, \tag{1}$$

the geometric equations

$$\varepsilon_{ik}(M, t) = \frac{1}{2}\left[U_{i,k}(M, t) + U_{k,i}(M, t)\right], \ M \in \mathscr{D}, \ t > 0, \tag{2}$$

the physical equations (Hooke's law)

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$$\sigma_{ik}(M, t) = 2\mu\varepsilon_{ik}(M, t) + \\ + \{\lambda\varepsilon_{kk}(M, t) - (3\lambda + 2\mu)\alpha_T[T(M, t) - T_0]\}\delta_{ik}, \; M \in \mathscr{D}, \; t > 0, \tag{3}$$

the compatibility equations

$$\gamma_{prm}\gamma_{qsn}\varepsilon_{rs,mn}(M, t) = 0, \tag{4}$$

where γ_{prm} is an antisymmetric tensor of third rank, and δ_{ik} is the Kronecker delta.

The relationships (1)-(4) result in the following general equation of dynamic thermoelasticity in stresses (in coordinate form)

$$(1+\nu)\Delta\sigma_{ik}(M, t) + \frac{\partial^2\sigma_\Sigma(M, t)}{\partial i\partial k} + \alpha_T E\left[\frac{\partial^2 T(M, t)}{\partial i\partial k} + \right. \\ \left. + \frac{1+\nu}{1-\nu}\Delta T(M, t)\delta_{ik}\right] = \frac{(1+\nu)\rho}{2G}\frac{\partial^2}{\partial t^2}\left[2\sigma_{ik}(M, t) - \right. \\ \left. - \frac{\nu}{1-\nu^2}\sigma_\Sigma(M, t)\delta_{ik} + \frac{2G(2+\nu)}{1-\nu}\alpha_T(T(M, t) - T_0)\delta_{ik}\right], \; M \in \mathscr{D}, \; t > 0, \tag{5}$$

where $\sigma_\Sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ is the sum of normal stresses, related to volume expansion $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ by the relationship

$$e(M, t) = \frac{1-2\nu}{E}\sigma_\Sigma(M, t) + 3\alpha_T[T(M, t) - T_0], \; M \in \mathscr{D}, \; t > 0. \tag{6}$$

Equation (5) is investigated in greatest detail in an examination of an elastic half-space $z > R$ of the temperature $T(z, t)$. The stresses occurring here will depend only on z and t, i.e., $\sigma_{ik} = \sigma_{ik}(z, t)$ and the displacements $U_x = U_y = 0$, $U_z = U_z(z, t)$. For this case (5) yields

$$\frac{\partial^2\sigma_{zz}}{\partial z^2} - \frac{1}{v_p^2}\frac{\partial^2\sigma_{zz}}{\partial t^2} = \frac{1+\nu}{1-\nu}\alpha_T\rho\frac{\partial^2 T(z, t)}{\partial t^2}, \; z > R, \; t > 0, \tag{7}$$

where $v_p = \sqrt{2G(1-\nu)/\rho(1-2\nu)} = \sqrt{(\lambda+2\mu)/\rho}$ is the sound speed in the material of the elastic domain.

Danilovskaya [12] first obtained (7) from relationships (1)-(3) and somewhat later Mura [13] (who apparently did not know about the earlier and more general paper [12]) did also independently. The solution of (7) under thermal and temperature heating conditions as well as of heating by a medium showed that the process of stress propagation is not purely diffusionary but is associated with elastic wave propagation [5].

We now pose the problem: Find a relationship analogous to (7) for a viscoelastic material and examine appropriate dynamic thermoviscoelasticity problems.

To formulate the rheological laws connecting the stress and strain, we introduce the stress $S_{ik}(M, t)$ and strain $e_{ik}(M, t)$ deviators by using the relationships [14]

$$S_{ik}(M, t) = \sigma_{ik}(M, t) - \sigma(M, t)\delta_{ik}, \tag{8}$$

$$e_{ik}(M, t) = \varepsilon_{ik}(M, t) - \varepsilon(M, t)\delta_{ik}, \tag{9}$$

where σ and ε are the mean normal stress and mean elongation:

$$\sigma(M, t) = \frac{1}{3}\sum_i \sigma_{ii}(M, t); \; \varepsilon(M, t) = \frac{1}{3}\sum_i \varepsilon_{ii}(M, t). \tag{10}$$

The relationship (6) in deviator form is

$$\varepsilon(M, t) = \frac{1-2\nu}{E}\sigma(M, t) + \alpha_T[T(M, t) - T_0]. \tag{11}$$

It remains valid even for viscoelastic bodies [14], which means that under hydrostatic compression or tension the body will behave as fully elastic.

Within the framework of a Maxwell medium, the dependence between the stress and strain for a viscoelastic body in deviator form will be

$$\frac{\partial S_{ik}(M,\ t)}{\partial t}+\frac{1}{\tau_p}S_{ik}(M,\ t)=2G\,\frac{\partial e_{ik}(M,\ t)}{\partial t},\ M\in\mathscr{D},\ t>0, \tag{12}$$

where the constant τ_p is the relaxation time of the medium: $\tau_p = \eta/G$.

Let us utilize relationships (1), (2), and (12) to derive the equation of dynamic thermo-viscoelasticity. As above, we consider a viscoelastic half-space z > R of temperature T(z, t). Here $U_x = U_y = 0$; $\varepsilon_{xx} = \varepsilon_{yy} = 0$; $e_{zz} = (2/3)\varepsilon_{zz}(x, t)$. Furthermore, we have

$$S_{zz}(z,\ t)=\sigma_{zz}(z,\ t)-\sigma(z,\ t), \tag{13}$$

$$\sigma(z,\ t)=\frac{E}{3(1-2\nu)}\varepsilon_{zz}(z,\ t)-\frac{E}{1-2\nu}\alpha_T[T(z,\ t)-T_0], \tag{14}$$

$$\frac{\partial S_{zz}}{\partial t}+\frac{1}{\tau_p}S_{zz}=\frac{4G}{3}\,\frac{\partial\varepsilon_{zz}(z,\ t)}{\partial t},\ t>0,\ \ S_{zz}(z,\ t)|_{t=0}=0, \tag{15}$$

$$\frac{\partial\sigma_{zz}}{\partial z}=\rho\frac{\partial^2U_z}{\partial t^2},\ z>R,\ t>0,$$

from which

$$\frac{\partial^2\sigma_{zz}}{\partial z^2}=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial U_z}{\partial z}\right)=\rho\frac{\partial^2\varepsilon_{zz}}{\partial t^2},\ z>R,\ t>0. \tag{16}$$

We find S_{zz} from (15) and later eliminate S_{zz} by using (13) and (14) and express ε_{zz} in terms of σ_{zz}, and by substituting the expression found for ε_{zz} into (16), we arrive at the desired dynamic thermoviscoelasticity equation in the form

$$\begin{aligned}\frac{\partial^2\sigma_{zz}}{\partial z^2}-\frac{1}{v_p^2}\frac{\partial^2\sigma_{zz}}{\partial t^2}&=\frac{1+\nu}{1-\nu}\alpha_T\rho\frac{\partial^2T}{\partial t^2}+\\&+\frac{m_1}{v_p^2\tau_p}\frac{\partial^2}{\partial t^2}\int_0^t\exp[-(m_2/3\tau_p)(t-\tau)]\,\sigma_{zz}(z,\ \tau)\,d\tau+\\&+\frac{m_1m_2}{\rho^{-1}\tau_p}\frac{\partial^2}{\partial t^2}\int_0^t\exp[-(m_2/3\tau_p)(t-\tau)]\,\alpha_T[T(z,\ \tau)-T_0]\,d\tau,\ z>R,\ t>0.\end{aligned} \tag{17}$$

Here $m_1 = 2(1 - 2\nu)/[3(1 - \nu)]$ and $m_2 = (1 + \nu)/(1 - \nu)$.

In the case of an elastic medium the relaxation time is $\tau_p = \infty$ ($\eta = \infty$), and (17) goes over into the Danilovskaya equation (7), therefore extending (7) to a viscoelastic body. The relationship (7) can be written in an even more compact form by going over to dimensionless variables

$$\begin{gathered}z'=z/R;\ \mathrm{Fo}=at/R^2;\ \alpha_0=v_pR/a;\ S=\alpha_T(3\lambda+2\mu)=\alpha_TE/(1-2\nu);\\ \mathrm{Fo}^{(r)}=a\tau_p/R^2;\ \beta_1=\frac{4\mu}{3\,\mathrm{Fo}^{(r)}(\lambda+2\mu)};\ \ \beta_2=\frac{3\lambda+2\mu}{3\,\mathrm{Fo}^{(r)}(\lambda+2\mu)};\\ \lambda=\frac{2G\nu}{1-2\nu};\ \mu=G=\frac{E}{2(1+\nu)};\ \sigma_{z'z'}(z',\ \mathrm{Fo})=\sigma_{zz}(z,\ t)/S(T_c-T_0);\\ T(z',\ \mathrm{Fo})=(T(z,\ t)-T_0)/(T_c-T_0);\ \alpha_0^2\frac{\partial^2\sigma_{z'z'}}{\partial z'^2}-\frac{\partial^2\sigma_{z'z'}}{\partial\,\mathrm{Fo}^2}=\frac{\partial^2T}{\partial\,\mathrm{Fo}^2}+\\ +\beta_1\frac{\partial^2}{\partial\mathrm{Fo}^2}\int_0^{\mathrm{Fo}}\exp[-\beta_2(\mathrm{Fo}-\tau)]\,[\sigma_{z'z'}(z',\ \tau)+T(z',\ \tau)]\,d\tau,\ z'>1,\ \mathrm{Fo}>0.\end{gathered} \tag{18}$$

More general problems can be formulated for (18), including the simultaneous examination of elastic and viscoelastic media. The circle of problems occurring here is sufficiently extensive and also includes the problems: computation of the magnitudes of the stress jumps on a thermoviscoelastic wave front in terms of external heating functions; estimation of the time of action of inertial effects within the framework of the model representations (15)-(18); the effect of a finite heat propagation rate; the effect of the influence of the motion velocity of the body boundary surface ($z > R + vt$, $t > 0$, $v = \text{const}$) on the magnitude of the thermoviscoelastic stresses; the effect of connectedness of the strain and temperature fields in the differential equation of heat conduction [5]; the effect of a finite heating rate of the boundary surface in a study of thermal impact under temperature heating conditions of the form [5]

$$T(z,\ t)|_{z=R} = \frac{T_S - T_0}{t_0}[t - \eta(t - t_0)(t - t_0)],\ t > 0,$$

where $\eta(z)$ is the Heaviside function, T_S is the surface temperature, t_0 is the time to reach the boundary surface $z = R$ of temperature T_S, as well as other questions whose solution within the framework of the phenomenological models for (18) is of great practical interest for the development of a theory of nonisothermal fracture.

As an illustration of (18), we examine the thermal reaction of a viscoelastic half-space $z > R$ (free from external loads) to a thermal impact produced by a medium of temperature T_C

$$\frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial^2 T}{\partial z'^2},\ z' > 1,\ \mathrm{Fo} > 0; \tag{19}$$

$$T(z',\ \mathrm{Fo})|_{\mathrm{Fo}=0} = 0,\ z' > 1; \tag{20}$$

$$\left.\frac{\partial T(z',\ \mathrm{Fo})}{\partial z'}\right|_{z'=1} = \mathrm{Bi}\,[T(z',\ \mathrm{Fo})|_{z'=1} - 1],\ \mathrm{Fo} > 0; \tag{21}$$

$$|T(z',\ \mathrm{Fo})| < \infty,\ z' > 1. \tag{22}$$

Here $T(z', \mathrm{Fo}) = [T(z, t) - T_0]/(T_C - T_0)$ and $\mathrm{Bi} = h/R$. Hence,

$$\sigma_{z'z'}(z',\ \mathrm{Fo})|_{\mathrm{Fo}=0} = \left.\frac{\partial \sigma_{z'z'}(z',\ \mathrm{Fo})}{\partial \mathrm{Fo}}\right|_{\mathrm{Fo}=0} = 0,\ z > 1, \tag{23}$$

$$\sigma_{z'z'}(z',\ \mathrm{Fo})|_{z'=1} = \sigma_{z'z'}(z',\ \mathrm{Fo})|_{z'=\infty} = 0,\ \mathrm{Fo} > 0. \tag{24}$$

The function $T(z', \mathrm{Fo})$ has the form [7, 15]

$$T(z',\ \mathrm{Fo}) = \Phi^*\left(\frac{z'-1}{2\sqrt{\mathrm{Fo}}}\right) - \exp[\mathrm{Bi}^2\mathrm{Fo} + \mathrm{Bi}(z'-1)]\,\Phi^*\left(\frac{z'-1}{2\sqrt{\mathrm{Fo}}} + \mathrm{Bi}\sqrt{\mathrm{Fo}}\right), \tag{25}$$

and in transform space [according to Laplace $L(T) = \bar{T}(z', p) = \int_0^\infty \exp(-p\mathrm{Fo})T(z', \mathrm{Fo})d\mathrm{Fo}$]

$$\bar{T}(z',\ p) = \frac{1}{p(1 + \sqrt{p}/\mathrm{Bi})}\exp[-(z'-1)\sqrt{p}]. \tag{26}$$

Here $\Phi^*(x) = 1 - \Phi(x)$; $\Phi(x) = (2/\sqrt{\pi})\int_0^x \exp(-y^2)dy$ is the Laplace function. The operational solution of (18) with the conditions (23) and (24) and with (26) taken into account has the form

$$\bar{\sigma}_{z'z'}(z',\ p) = \frac{p + \beta_1 + \beta_2}{(1 + \sqrt{p}/\mathrm{Bi})\,[p^2 - (\alpha_0^2 - \beta_1 - \beta_2)\,p - \alpha_0^2\beta_2]} \times$$
$$\times \left\{\exp\left[-\frac{z'-1}{\alpha_0}\,p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right] - \exp[-(z'-1)\sqrt{p}]\right\}. \tag{27}$$

Taking into account that expressions of the type (27) are characteristic for dynamic problems of the form (18)-(24) and finding the original for the transform (27) represents definite technical difficulties, we clarify the procedure for going over to $\sigma_{z'z'}(z', \mathrm{Fo})$.

First, we expand the preexponential factor into the sum of fractions

$$\frac{p+\beta_1+\beta_2}{p^2-(\alpha_0^2-\beta_1-\beta_2)\,p-\alpha_0^2\beta_2}=\sum_{i=1}^{2}\frac{A_i}{p-\gamma_i}, \tag{28}$$

where

$$\gamma_i=\frac{\alpha_0^2-\beta_1-\beta_2+(-1)^{i-1}\sqrt{(\alpha_0^2-\beta_1-\beta_2)^2+4\alpha_0^2\beta_2}}{2},$$

$$A_i=\frac{\gamma_i+\beta_1+\beta_2}{(-1)^{i-1}(\gamma_1-\gamma_2)},\quad i=1,\ 2. \tag{29}$$

It follows from (29) that $\gamma_1 > 0$ and $\gamma_2 < 0$; moreover, $-(\beta_1+\beta_2) < \gamma_2 < -\beta_2$; $A_1 > 0$, $A_2 < 0$, $A_1 + A_2 = 1$. Let us note that for an elastic medium ($\beta_1 = \beta_2 = 0$) there follows $\gamma_1 = \alpha_0^2$, $\gamma_2 = 0$, $A_1 = 1$, $A_2 = 0$.

Taking account of (29) we find

$$L^{-1}\left\{\overline{\Theta}(p)=\frac{p+\beta_1+\beta_2}{(1+\sqrt{p}/\mathrm{Bi})\,[p^2-(\alpha_0^2-\beta_1-\beta_2)\,p-\alpha_0^2\beta_2]}\right\}=$$
$$=\sum_{i=1}^{2}\frac{\mathrm{Bi}^2}{\mathrm{Bi}^2-\gamma_i}\Bigg[\exp(\gamma_i\,\mathrm{Fo})-\exp(\mathrm{Bi}^2\mathrm{Fo})\,\Phi^*(\mathrm{Bi}\sqrt{\mathrm{Fo}})-$$
$$-\frac{\gamma_i}{\mathrm{Bi}\sqrt{\pi}}\exp(\gamma_i\,\mathrm{Fo})\int_0^{\mathrm{Fo}}\frac{1}{\sqrt{\tau}}\exp(-\gamma_i\tau)\,d\tau\Bigg]. \tag{30}$$

The key question is finding the original for the transform

$$\exp\left[-\frac{z'-1}{\alpha_0}\,p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right]. \tag{31}$$

Initially we find the original of the transform

$$\overline{Q}(p)=\frac{1}{p}\exp\left[-\frac{z'-1}{\alpha_0}\,p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right], \tag{32}$$

by applying the contour displayed in Fig. 1 for the calculation of the Riemann–Mellin integral $(1/2\pi i)\int_{\gamma-i\infty}^{\gamma+i\infty}\exp(p\mathrm{Fo})\overline{\varphi}(p)dp$. We find

$$L^{-1}[\overline{Q}(p)]=L^{-1}\left\{\frac{1}{p}\exp\left[-\frac{z'-1}{\alpha_0}\,p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right]\right\}=Q(\mathrm{Fo})=$$
$$=\eta\left(\mathrm{Fo}-\frac{z'-1}{\alpha_0}\right)\Bigg[1-\frac{1}{\pi}\int_0^{\beta_1}\frac{1}{x+\beta_2}\times$$
$$\times\exp[-(x+\beta_2)\,\mathrm{Fo}]\sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right)dx\Bigg]. \tag{33}$$

from which there follows

$$Q(\mathrm{Fo})=\begin{cases}Q(\mathrm{Fo}), & \mathrm{Fo}>\dfrac{z'-1}{\alpha_0},\\[2mm] 0, & \mathrm{Fo}<\dfrac{z'-1}{\alpha_0}.\end{cases} \tag{34}$$

The function Q(Fo) allows a jump in going through the value $\mathrm{Fo} = (z' - 1)/\alpha_0$. The magnitude of this jump equals

$$|\Delta|=\lim_{\mathrm{Fo}\to\frac{z'-1}{\alpha_0}+0}Q(\mathrm{Fo})=\lim_{y\to 0+}Q\left(y+\frac{z'-1}{\alpha_0}\right)=Q\left(\frac{z'-1}{\alpha_0}+0\right)=$$

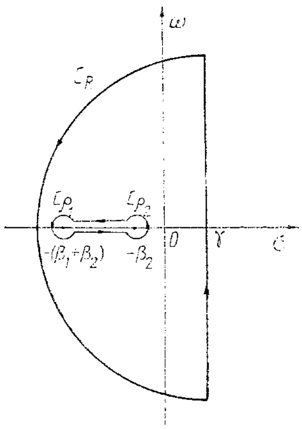

Fig. 1. Contour for finding the original for the transform (32) by using the Riemann–Mellin integral.

$$= 1 - \frac{1}{\pi} \int_0^{\beta_1} \frac{1}{x+\beta_2} \exp\left[-(x+\beta_2)\frac{z'-1}{\alpha_0}\right] \times \sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1 - x}{x}}\right) dx. \tag{35}$$

Let us compute the same quantity $|\Delta|$ by using an operational approach. First, we note that an initial value theorem must be formulated for functions of the type (34) by taking account of its absence in known operational calculus handbooks [7, 16-18].

THEOREM (on the initial value). If $\lim\limits_{y\to 0+} Q\left(y + \frac{z'-1}{\alpha_0}\right)$ exists, then

$$\lim_{y\to 0+} Q\left(y + \frac{z'-1}{\alpha_0}\right) = \lim_{p\to\infty} p\bar{Q}(p)\exp\left(\frac{z'-1}{\alpha_0}p\right), \tag{36}$$

where the variable p running through real values tends to $+\infty$. This property is conserved even in the case when $p \to \infty$ along a ray of the complex plane making an angle less than $\pi/2$ in absolute value with the positive real axis.

Proof. We have

$$\bar{Q}(p) = \int_0^\infty \exp(-p\mathrm{Fo})Q(\mathrm{Fo})\,d\,\mathrm{Fo} = \int_{(z'-1)/\alpha_0}^\infty \exp(-p\,\mathrm{Fo})\,Q(\mathrm{Fo})\,d\,\mathrm{Fo} =$$

$$= \exp\left(-\frac{z'-1}{\alpha_0}p\right)\int_0^\infty \exp(-py)\,Q\left(y + \frac{z'-1}{\alpha_0}\right)dy,$$

from which

$$\bar{Q}(p)\exp\left(\frac{z'-1}{\alpha_0}p\right) = \int_0^\infty \exp(-py)\,Q\left(y + \frac{z'-1}{\alpha_0}\right)dy.$$

Going over to the variable u = py in the integral on the right, we obtain

$$p\bar{Q}(p)\exp\left(\frac{z'-1}{\alpha_0}p\right) = \int_0^\infty \exp(-u)\,Q\left(\frac{u}{p} + \frac{z'-1}{\alpha_0}\right)du.$$

Passing to the limit as $p \to \infty$, we find

$$\lim_{p\to\infty} p\bar{Q}(p)\exp\left(\frac{z'-1}{\alpha_0}p\right) = Q\left(\frac{z'-1}{\alpha_0} + 0\right) = |\Delta|. \tag{37}$$

Relationships (32) and (37) result in

$$|\Delta| = \lim_{p\to\infty} p\bar{Q}(p)\exp\left(\frac{z'-1}{\alpha_0}p\right) =$$

$$= \lim_{p\to\infty} \exp\left[\frac{z'-1}{\alpha_0}\left(p - p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right)\right] = \exp\left[\frac{\beta_1(z-1)}{2\alpha_0}\right]. \quad (38)$$

Comparing (35) and (38), we obtain the integral

$$\exp\left[-\frac{\beta_1(z'-1)}{2\alpha_0}\right] = = 1 - \frac{1}{\pi}\int_0^{\beta_1}\frac{\exp\left[-(x+\beta_2)\dfrac{z'-1}{\alpha_0}\right]}{x+\beta_2}\sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right)dx, \quad (39)$$

which is important for further manipulations and is also of independent interest. We find the desired original of the transform (31)

$$L^{-1}\left[\exp\left(-\frac{z'-1}{\alpha_0}p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right)\right] = = \delta\left(\mathrm{Fo} - \frac{z'-1}{\alpha_0}\right)\left[1 - \frac{1}{\pi}\int_0^{\beta_1}\frac{1}{x+\beta_2}\exp(-(x+\beta_2)\mathrm{Fo}) \times \times \sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right)dx\right] + + \eta\left(\mathrm{Fo} - \frac{z'-1}{\alpha_0}\right)\frac{1}{\pi}\int_0^{\beta_1}\exp(-(x+\beta_2)) \times \times \sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right)dx \quad (40)$$

from (33) by the rule of differentiating the original $\int_0^{\mathrm{Fo}} f(\tau)\,d\tau \leftarrow \frac{1}{p}\bar{f}(p)$ where $\delta(z)$ is the Dirac delta function. Relationships (30), (39), and (40) obtained permit writing the original for $\sigma_{z'z'}(z', \mathrm{Fo})$ for the transform (27). We find

$$\sigma_{z'z'}(z', \mathrm{Fo}) = \sigma^{(1)}_{z'z'}(z', \mathrm{Fo}) + \begin{cases} 0, & \mathrm{Fo} < \dfrac{z'-1}{\alpha_0} \\ \sigma^{(2)}_{z'z'}(z', \mathrm{Fo}), & \mathrm{Fo} > \dfrac{z'-1}{\alpha_0}; \end{cases} \quad (41)$$

$$\sigma^{(1)}_{z'z'}(z', \mathrm{Fo}) = \sum_{i=1}^{2}\frac{A_i\mathrm{Bi}^2}{\mathrm{Bi}^2-\gamma_i}\left[\exp(\mathrm{Bi}^2\mathrm{Fo} + \mathrm{Bi}(z'-1))\,\Phi^*\left(\frac{z'-1}{2\sqrt{\mathrm{Fo}}} + + \mathrm{Bi}\sqrt{\mathrm{Fo}}\right) - \exp(\gamma_i\mathrm{Fo})\left(\frac{z'-1}{2\sqrt{\pi}}\int_0^{\mathrm{Fo}}\frac{\exp\left(\gamma_i\tau - \dfrac{(z'-1)^2}{4\tau}\right)}{\tau^{3/2}}d\tau - - \frac{\gamma_i}{\mathrm{Bi}\sqrt{\pi}}\int_0^{\mathrm{Fo}}\frac{\exp\left(-\gamma_i\tau - \dfrac{(z'-1)^2}{4\tau}\right)}{\sqrt{\tau}}d\tau\right)\right];$$

$$\sigma^{(2)}_{z'z'}(z', \mathrm{Fo}) = \sum_{i=1}^{2}\frac{A_i\mathrm{Bi}^2}{\mathrm{Bi}^2-\gamma_i}\left\{\exp\left(-\frac{\beta_1(z'-1)}{2\alpha_0}\right)\left[\exp\left(\gamma_i\left(\mathrm{Fo} - \frac{z'-1}{\alpha_0}\right)\right) \times \times \left(1 - \frac{\gamma_i}{\mathrm{Bi}\sqrt{\pi}}\int_0^{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\frac{\exp(-\gamma_i\tau)}{\sqrt{\tau}}d\tau\right) - - \exp\left(\mathrm{Bi}^2\left(\mathrm{Fo} - \frac{z'-1}{\alpha_0}\right)\right)\Phi^*\left(\mathrm{Bi}\sqrt{\mathrm{Fo} - \frac{z'-1}{\alpha_0}}\right)\right] -$$

$$-\frac{1}{\pi}\int_0^{\beta_1}\exp(-(x+\beta_2)\,\mathrm{Fo})\sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right)\times$$

$$\times\left[\frac{1}{x+\beta_2+\gamma_i}\left(\exp\left[(x+\beta_2+\gamma_i)\left(\mathrm{Fo}-\frac{z'-1}{\alpha_0}\right)\right]\times\right.\right.$$

$$\times\left(1-\frac{\gamma_i}{\mathrm{Bi}\sqrt{\pi}}\int_0^{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\frac{1}{\sqrt{\tau}}\exp(-\gamma_i\tau)\,d\tau\right)-$$

$$-\left(1-\frac{\gamma_i}{\mathrm{Bi}\sqrt{\pi}}\int_0^{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\frac{1}{\sqrt{\tau}}\exp((x+\beta_2)\tau)d\tau\right)\Bigg)-\frac{1}{x+\beta_2+\mathrm{Bi}^2}\times$$

$$\times\left(\exp\left[(x+\beta_2+\mathrm{Bi}^2)\left(\mathrm{Fo}-\frac{z'-1}{\alpha_0}\right)\right]\Phi^*\left(\mathrm{Bi}\sqrt{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\right)-\right.$$

$$\left.\left.-\left(1-\frac{\mathrm{Bi}}{\sqrt{\pi}}\int_0^{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\frac{1}{\sqrt{\tau}}\exp((x+\beta_2)\tau)\,d\tau\right)\right)\right]dx\Bigg\}. \tag{42}$$

According to (29), for an elastic medium, $\gamma_1 = \alpha_0^2$, $A_1 = 1$, $A_2 = 0$, and the thermoelastic reaction of a solid under sudden heating by a medium is described by relationships (41), where

$$\sigma^{(1)}_{z'z'}(z',\ \mathrm{Fo}) = -\frac{1}{2(1+\alpha_0/\mathrm{Bi})}\exp(\alpha_0^2\mathrm{Fo}-$$

$$-\alpha_0(z'-1))\Phi^*\left(\frac{z'-1}{2\sqrt{\mathrm{Fo}}}-\alpha_0\sqrt{\mathrm{Fo}}\right)-$$

$$-\frac{1}{2(1-\alpha_0/\mathrm{Bi})}\exp(\alpha_0^2\,\mathrm{Fo}+\alpha_0(z'-1))\Phi^*\left(\frac{z'-1}{2\sqrt{\mathrm{Fo}}}+\alpha_0\sqrt{\mathrm{Fo}}\right)- \tag{43}$$

$$-\frac{\mathrm{Bi}^2}{\mathrm{Bi}^2-\alpha_0^2}\exp(\mathrm{Bi}^2\,\mathrm{Fo}+\mathrm{Bi}(z'-1))\Phi^*\left(\frac{z'-1}{2\sqrt{\mathrm{Fo}}}+\mathrm{Bi}\sqrt{\mathrm{Fo}}\right);$$

$$\sigma^{(2)}_{z'z'}(z',\ \mathrm{Fo}) = \frac{\mathrm{Bi}^2}{\mathrm{Bi}^2-\alpha_0^2}\left[\exp\left[\alpha_0^2\left(\mathrm{Fo}-\frac{z'-1}{\alpha_0}\right)\right]\left(1-\right.\right.$$

$$\left.-\frac{\alpha_0}{\mathrm{Bi}}\Phi\left(\alpha_0\sqrt{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\right)\right)-$$

$$\left.-\exp\left[\mathrm{Bi}^2\left(\mathrm{Fo}-\frac{z'-1}{\alpha_0}\right)\right]\Phi^*\left(\mathrm{Bi}\sqrt{\mathrm{Fo}-\frac{z'-1}{\alpha_0}}\right)\right]. \tag{44}$$

This result issues from (42) and agrees with that obtained in [5].

In the case of temperature heating in (21) (1/Bi = 0), the solution also has the form (41), where

$$\sigma^{(1)}_{z'z'}(z',\ \mathrm{Fo}) = \sum_{i=1}^{2}\frac{A_i(z'-1)}{2\sqrt{\pi}}\exp(\gamma_i\,\mathrm{Fo})\int_0^{\mathrm{Fo}}\frac{1}{\tau^{3/2}}\exp\left(-\gamma_i\tau-\frac{(z'-1)^2}{4\tau}\right)d\tau, \tag{45}$$

$$\sigma^{(2)}_{z'z'}(z',\ \mathrm{Fo}) = \sum_{i=1}^{2}A_i\left\{\exp\left[-\frac{\beta_1(z'-1)}{2\alpha_0}+\gamma_i\left(\mathrm{Fo}-\frac{z'-1}{\alpha_0}\right)\right]-\right.$$

$$-\frac{1}{\pi}\exp(\gamma_i\,\mathrm{Fo})\int_0^{\beta_1}\frac{1}{x+\beta_2+\gamma_i}\left[\exp\left(-(x+\beta_2+\gamma_i)\frac{z'-1}{\alpha_0}\right)-\right.$$

$$\left.\left.-\exp(-(x+\beta_2+\gamma_i)\,\mathrm{Fo})\right]\sin\left(\frac{z'-1}{\alpha_0}(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right)dx\right\}. \tag{46}$$

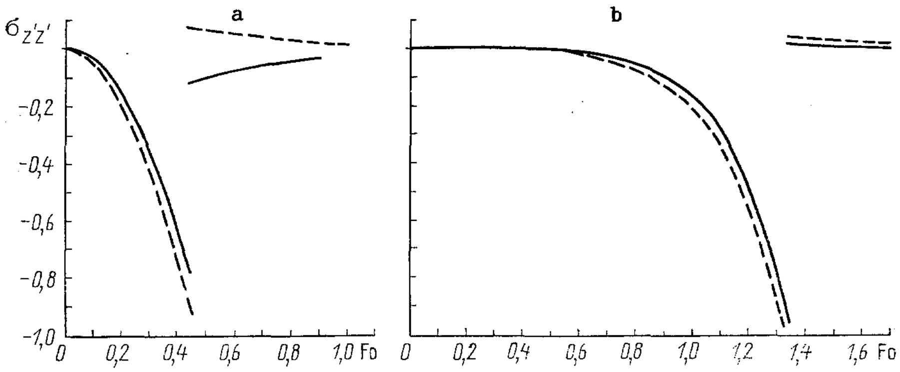

Fig. 2. Stress distribution with time in the section z' = 2 for $\eta = 5\cdot10^3$ Pa·sec (a) and in the section z' = 4 for $\eta = 5\cdot10^5$ Pa·sec (b); solid line is for a viscoelastic medium and the dashes for an elastic medium.

Let us analyze the results obtained.

It follows from (41) that the stress component expressed by the function $\sigma_{z'z'}^{(1)}(z', Fo)$ is a diffusion wave that occurs at once at each interior point of the viscoelastic domain; the stress component expressed by the function $\sigma_{z'z'}^{(2)}(z', Fo)$ is a longitudinal wave whose front moves at the velocity α_0 within the viscoelastic half-space. A stress described by function $\sigma_{z'z'}^{(1)}(z', Fo)$ that grows from zero to a certain negative value occurs initially at an arbitrary interior point (z', Fo) of the domain. At the time $Fo = (z' - 1)/\alpha_0$, a wave described by the function $\sigma_{z'z'}^{(2)}(z', Fo)$ arrives at this point and the stress $\sigma_{z'z'}(z', Fo)$ makes a jump by the quantity $|\Delta| = \exp[-\beta_1(z' - 1)/2\alpha_0)]$ into the domain of positive (tensile) values or, by remaining in the domain of negative values after the jump, decreases rapidly (as in the first case) to zero. The magnitude of the stress jump on the wave front hence decreases with advancement into the depths of the viscoelastic domain. The distribution of the stress $\sigma_{z'z'}(z', Fo)$ in the section z' = 2 is presented in Fig. 2a for $\alpha_0 = 2.26$, $\eta = 5\cdot10^3$ Pa·sec, $G = 3\cdot10^5$ Pa, $\nu = 0.28$, $\beta_1 = 1.9$, $\beta_2 = 2.78$, $\gamma_1 = 3.99$, and $\gamma_2 = -3.56$, computed for a viscoelastic medium (solid curve) by using (45) and (46) and for an elastic medium (dashes) by using (43) and (44). As is seen from the figure, short-range stresses whose magnitude differs essentially from the corresponding stresses for an elastic domain occur during sudden heating of the viscoelastic half-space boundary because of the action of inertia forces therein (in particular in the section z' = 2). This indicates the closeness of the nature of the thermal reactions of elastic and viscoelastic media to thermal impact. However, the influence of viscosity of the medium (the influence of relaxation processes occurring in the medium) becomes noticeable at the time of expansion wave passage since a change occurs in the maximum compressive stress toward diminution at the time of the stress jump and at subsequent times. It is essential that for comparatively small values of the viscosity η the stress $\sigma_{z'z'}$ should not emerge beyond the negative-values limit, remaining compressive during its whole time of variation. As the viscosity of the medium increases, the mentioned difference diminishes and becomes inessential, as the nature of the curves in Fig. 2b for $\eta = 5\cdot10^5$ Pa·sec, z' = 4 indicates (remain constant as in Fig. 2a).

As is seen from (27) and (37) for the case of heating by a medium (Bi > 0), no discontinuities are observed and the stresses vary continuously:

$$|\Delta| = \lim_{p\to\infty} p\bar{\Theta}(p)\exp\left[\frac{z'-1}{\alpha_0}\left(p - p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right)\right] = 0.$$

NOTATION

ρ, mass of unit volume; λ, μ, isothermal Lamé coefficients; α_T, coefficient of thermal expansion; E, G, the elastic and shear moduli; ν, the Poisson ratio; η, the viscosity coefficient; Bi = hR, the Biot criterion; h, the relative coefficient of heat transfer; $Fo = at/R^2$, the Fourier criterion; a, the thermal diffusivity of the material; L^{-1}, the inverse operator to the Laplace operator.

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ELECTROCONVECTON AND HEAT EXCHANGE IN DISPERSED GAS–LIQUID SYSTEMS

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The problem of momentum, energy, and electrical charge transport is formulated for gas–liquid dispersed systems; electroconvection, heat liberation, and interphase heat exchange are considered, and practical applications of such studies in diffuser systems are described.

Introduction. In a number of branches of industry, such as thermal energy production and chemical and food technology processes in which a gas interacts with a liquid are often used, and they are often carried out under bubbler conditions to intensify them. The main questions involved in study of such processes are the hydrodynamics of the gas–liquid layer, removal of the liquid phase, heat-mass transport and organization of various processes both in the bubble layer and the vapor–gas space.

The action of electric fields can significantly intensify heat exchange in gas–liquid media. To a certain extent such questions have been investigated in bubble boiling [1]. It follows from data available in the literature on the problem of heat exchange in bubble-

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